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THEORETICAL ELASTIC DEFORMATIONS OF A THICK HORIZONTAL CIRCULAR PLATE HAVING INTERRUPTED PERIPHERAL ARC SUPPORTS

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Abstract—In this paper solutions are determined for the deflection under its own weight, moments and shears in a thick, horizontally oriented, circular plate having interrupted peripheral arc supports. A theory developed by Reïssner for thick plates that includes shear deformations is used, and the results are compared with those of classical plate theory. It is found that for plates having thickness-to-diameter ratios greater than approximately one-tenth, shearing effects can contribute significantly to the total deflection and stresses, and hence should not be neglected. Numerical results are presented and interpreted in detail.

NOTATION

w	deflection
r	variable radius
R	outer radius of plate
h	thickness of plate
W	total weight of plate
ν	Poisson's ratio
E	Young's modulus
D	flexural rigidity $= Eh^3/12(1-\nu^2)$
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zz}$	stresses acting on element of plate
M_r	bending couples
$M_{r\theta}$	twisting couple
V_r, V_θ	transverse shear stress resultants
P	normal load acting on plate
θ	angle
P_0	a constant $= -W/\pi R^2$
k^2	a constant $= \left(\frac{1-\nu}{2}\right)\frac{D}{C_s}$
G_m, A_m, E_m	coefficients appearing in expressions for deflection, moments and shears
S	shear stress acting on support arc
l	one-half support arc (rad.)
G	shear modulus $= \frac{E}{2(1+\nu)}$
n	number of support arcs
∇^2	$\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}$
C_s, C_n	constants depending on the properties of the material and the nature of the stress distribution across the thickness of the plate

INTRODUCTION

SCIENTISTS and engineers have long been concerned about the deflection of plates under various support configurations. Couder [1] gives the solution for the deflection of a flat, back-supported cylindrical plate. The effects of various supporting configurations on the deflections of a vertically oriented mirror are studied by Schwesinger [2]. Bassali [3] determines expressions for the deflection of a thin circular plate, supported at several interior or boundary points. Kirstein and Woolley [4, 5] reduce Bassali's solution to the case of symmetrical bending. A circular mirror having a linearly varying thickness, supported along a central hole, and free along the outer edge is analyzed by Prevenslik [6]. Shear deflections have not been considered in any of these papers.

Malvick and Pearson [7] determine the elastic deflection, including shear effects, of a solid quartz mirror with a large central hole, flat back and a spherically dished front surface. They solve the three-dimensional elasticity equations by a dynamic relaxation technique programmed for a large-scale digital computer.

Reissner [8, 9] derives a system of equations for the bending of elastic plates, which takes into account the transverse shear deformability of the plate.

In this paper, Reissner's theory is applied to the problem of a horizontally oriented, thick circular plate having interrupted peripheral arc supports. Solutions are obtained for the deflection under its own weight, the moments and shears in the plate; results are compared with those obtained from classical bending theory. The effects of shear contributions to the total deformation and stresses are illustrated. Numerical results are presented and discussed in detail.

ANALYSIS

Consider a horizontally oriented, circular plate supported by n support arcs about its periphery. Figure 1 shows the angular location of the center of the support arcs for the general case of n supports. The center of one support arc is located at $r = R$, $\theta = 0°$.

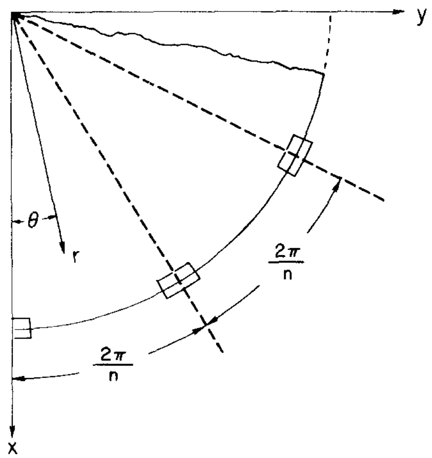

FIG. 1. Angular location of support arcs for the general case of n supports.

Expressions for the deflection under its own weight, moments and shears in the disk are desired. Figure 2 shows the stresses acting on an infinitesimal element of plate material. Let M_r be the bending couple, $M_{r\theta}$ the twisting couple and V_r and V_θ the transverse

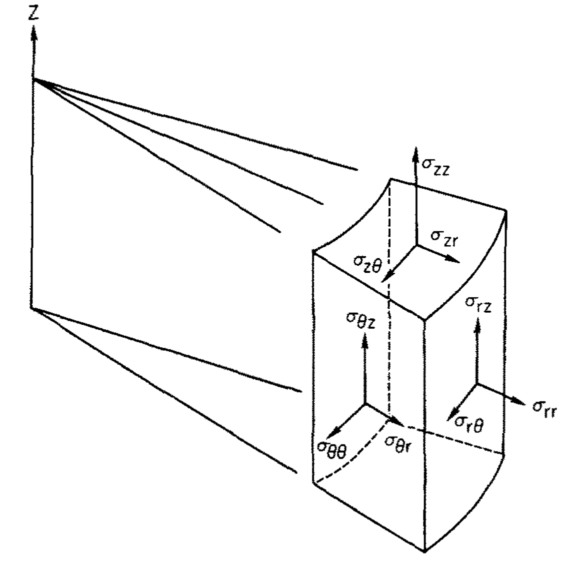

FIG. 2. Stresses acting on element of disk.

shear-stress resultants; these are expressed in terms of the stresses as follows:

$$
\begin{aligned}
M_r &= \int_{-h/2}^{h/2} z\sigma_{rr}\,\mathrm{d}z, \qquad & M_{r\theta} &= \int_{-h/2}^{h/2} z\sigma_{r\theta}\,\mathrm{d}z, \\
V_r &= \int_{-h/z}^{h/2} \sigma_{rz}\,\mathrm{d}z, \qquad & V_\theta &= \int_{-h/2}^{h/2} \sigma_{\theta z}\,\mathrm{d}z.
\end{aligned}
\tag{1}
$$

Reissner [9] gives the generalized stress–strain relations of the problem as:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = -P, \tag{2a}$$

$$V_r - k^2\left(\nabla^2 V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r^2}\right) = -\frac{\partial H}{\partial r}, \tag{2b}$$

$$V_\theta - k^2\left(\nabla^2 V + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r^2}\right) = -\frac{1}{r}\frac{\partial H}{\partial \theta}, \tag{2c}$$

where

$$H = D\nabla^2 w + D(1+\nu)\left(\frac{1}{2C_s}-\frac{1}{C_n}\right)P. \tag{2d}$$

The equations for the bending and twisting moments, also from Reissner [9], are:

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}+\frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right]+2k^2\frac{\partial V_r}{\partial r}-D\left(\frac{1+\nu}{C_n}-\frac{\nu}{C_s}\right)P, \tag{2e}$$

$$M_{r\theta} = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)+k^2\left[\frac{1}{r}\frac{\partial V_r}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right]. \tag{2f}$$

Following the procedure outlined by Reissner [8], it can be shown that the solutions for w, V_r and V_θ, continuous and finite at the origin of coordinates, which yield finite moments and slopes at the origin, are:

$$Dw = \sum_{m=0}^{\infty}\left[G_m r^m - \frac{A_m}{4(m+1)}r^{m+2}\right]\cos m\theta + \frac{r^2}{4}\left\{\frac{r^2}{16} - \left[k^2 + D(1+\nu)\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\right]\right\}P_0, \tag{3a}$$

$$V_r = \sum_{m=1}^{\infty}\left[mA_m r^{m-1} + \frac{m}{r}E_m I_m\left(\frac{r}{k}\right)\right]\cos m\theta - \frac{P_0 r}{2}, \tag{3b}$$

$$V_\theta = -\sum_{m=1}^{\infty}\left\{mA_m r^{m-1} + E_m\left[\frac{m}{r}I_m(r/k) + \frac{1}{k}I_{m+1}(r/k)\right]\right\}\sin m\theta. \tag{3c}$$

In equations (3a)–(3c), it is assumed that

$$P(r,\theta) = P_0 = \frac{-W}{\pi R^2}. \tag{3d}$$

Also, $I_m(r/k)$ and $I_{m+1}(r/k)$ are modified Bessel functions of the first kind of argument (r/k) and order m and $m+1$, respectively.

The expressions for the bending and twisting moments become:

$$\begin{aligned} M_r = \sum_{m=0}^{\infty}\cos m\theta\Bigg\{&A_m\left[2k^2 m(m-1)r^{m-2} + \frac{m^2(1-\nu)+m(3+\nu)+2(\nu+1)}{4(m+1)}r^m\right] \\ &-(1-\nu)m(m-1)G_m r^{m-2} + \frac{2k^2}{r^2}mE_m\left[(m-1)I_m(r/k) + \frac{r}{k}I_{m+1}(r/k)\right]\Bigg\} \\ &+P_0\left[\frac{D(1-\nu^2)}{2C_n} - \frac{(3+\nu)r^2}{16}\right], \end{aligned} \tag{3e}$$

$$\begin{aligned} M_{r\theta} = \sum_{m=1}^{\infty}\sin m\theta\Bigg\{&A_m\left[\frac{(\nu-1)m}{4}r^m - 2m(m-1)k^2 r^{m-2}\right] + (1-\nu)m(m-1)r^{m-2}G_m \\ &+E_m\left[-I_m(r/k) - \frac{2k^2}{r^2}m(m-1)I_m(r/k) + \frac{2k}{r}I_{m+1}(r/k)\right]\Bigg\}. \end{aligned} \tag{3f}$$

The expressions given by equations (3a)–(3f) must now be solved subject to the boundary conditions on the variables V_r, M_r and $M_{r\theta}$.

To this end, the stress components in the regions of support are defined as follows:

$$\sigma_{rr} = 0, \tag{4a}$$

$$\sigma_{r\theta} = 0, \tag{4b}$$

$$\sigma_{rz} = S. \tag{4c}$$

Summation of forces in the z direction yields:

$$2nlRhS - W = 0, \tag{5a}$$

$$S = \frac{W}{2nlRh}. \tag{5b}$$

The quantities M_r, $M_{r\theta}$ and V_r may next be expressed on the boundary as follows:

$$M_r(R, \theta) = 0, \tag{6a}$$

$$M_{r\theta}(R, \theta) = 0, \tag{6b}$$

$$V_r(R, \theta) = Sh\left(\frac{nl}{\pi} + \sum_{m=1}^{\infty} \alpha_m \cos m\theta\right), \tag{6c}$$

where

$$\alpha_m = \frac{2}{m\pi} \sin ml \sum_{s=1}^{n} \cos(s-1)\frac{2m\pi}{n} m \geq 1. \tag{6d}$$

Because of the existence of arc supports at n equally spaced arcs, non-zero values of α_m are obtained for only those values of m that are integral multiples of n.

Comparison is next made between equations (3e), (3f), and (3b) with (6a)–(6c), respectively; for $m = 0$, this yields:

$$A_0 = \frac{2}{(1+\nu)}\left\{\frac{(3+\nu)R^2}{16} - \frac{D(1-\nu^2)}{2C_n}\right\}P_0, \tag{7a}$$

$$S = \frac{-P_0\pi R}{2nlh} \qquad m = 0. \tag{7b}$$

The general expressions for the coefficients for $m \geq 2$ become:

$$A_m = -\frac{S\alpha_m h}{\Delta_m}\{2k(1-\nu)(m^2-1)mR^{(m-3)}I_{m+1} - m(m-1)(1-\nu)R^{(m-2)}I_m\}, \tag{8a}$$

$$E_m = \frac{S\alpha_m h}{\Delta_m}\left\{\frac{R^{(2m-2)}}{2} m(m-1)(1-\nu^2)\right\}, \tag{8b}$$

$$G_m = -\frac{S\alpha_m h}{\Delta_m}\left\{\left[\left(\frac{\nu(m-2)}{4} - \frac{(m+2)}{4}\right)R^m + (1-m)(3+\nu)mk^2R^{(m-2)}\right]I_m\right.$$
$$\left. + \left[\frac{k}{2}(m(m+1)(1-\nu)+2(1+\nu))R^{(m-1)} + 4m(m^2-1)k^3R^{(m-3)}\right]I_{m+1}\right\}, \tag{8c}$$

where

$$\Delta_m = (1-\nu)m^2(m-1)R^{(2m-3)}\left\{\frac{(\nu+3)}{2}I_m - \frac{2k}{R}(m+1)I_{m+1}\right\}. \tag{8d}$$

In equations (8a)–(8d), I_m and I_{m+1} are modified Bessel functions of the first kind of argument (R/k) and order m and $m+1$, respectively.

Because of the symmetrical arrangement of the support arcs, the general coefficients A_m, E_m and G_m have non-zero values for only those values of m that are integral multiples of n. It is unnecessary to determine A_1, E_1 and G_1, since results for the case in which $n = 1$ for which the plate is simply supported about its entire periphery may be obtained by choosing an integral value of $n > 1$, and l such that the product $nl = \pi$.

Reissner [9] shows that for solid plates of uniform thickness, C_n and C_s may be expressed as follows:

$$C_n = \frac{5}{6}\frac{Eh}{\nu}, \tag{9a}$$

$$C_s = \frac{5}{6}Gh. \tag{9b}$$

If $nl = \pi$ in the above expressions for deflections, the case is obtained in which the plate is simply supported about its entire periphery. The expression for deflection thus becomes:

$$w = \frac{W(r^2-R^2)}{64\pi R^2 D}\left[\frac{(5+\nu)}{(1+\nu)}R^2 - r^2\right] + \frac{W(r^2-R^2)}{8\pi D}\left[\frac{2}{5}\frac{h^2}{1-\nu^2}\right]. \tag{10a}$$

The first term appearing in equation (10a) represents the contribution to the total deflection due to bending, and the second term represents the shear effects. The bending expression is identical with that given by Timoshenko [10] for the case of a thin plate simply supported about its entire periphery.

The exact expression for the shear deflection of a circular plate simply supported about its periphery is given by Love [11]; his result is as follows:

$$w_{\text{shear}} = \frac{W(r^2-R^2)}{8\pi D}\left[\frac{2}{5}h^2\frac{(1+\nu/8+\nu^2/8)}{(1-\nu^2)}\right]. \tag{10b}$$

For $\nu = 0{\cdot}16$, the difference in shear correction terms between equations (10a) and (10b) is about 2 per cent; for $\nu = 0{\cdot}3$, this difference is about 4 per cent.

For comparison, the problem considered in this paper was solved using the equations of classical plate theory; the results for the deflection are as follows:

$$\begin{aligned} w = {} & \frac{WR^2}{64\pi D}\left[\left(\frac{r}{R}\right)^4 - \frac{2(3+\nu)}{(1+\nu)}\left(\frac{r}{R}\right)^2\right] \\ & - \sum_{m=1}^{\infty}\left\{\frac{nm-2(\nu+1)/(\nu-1)}{nm(nm-1)} - \left(\frac{r}{R}\right)^2\frac{1}{(nm+1)}\right\}\cdot \\ & \left\{\frac{R^2(r/R)^{nm}W}{2\pi n^2m^2l(\nu+3)D}\right\}\sin(nml)\cos(nm\theta). \end{aligned} \tag{11}$$

If $nl = \pi$ in equation (11), the results reduce exactly to those given by Timoshenko [10] for the case of a thin plate simply supported about its entire periphery.

NUMERICAL RESULTS

The plate is assumed to be fabricated of fused quartz, and to have the following properties: Young's modulus = $10{\cdot}6 \times 10^7$ lb/in.^{2}; Poisson's ratio = 0·165; density = 0·079 lb/in.^{3} The outer radius is assumed to be 6 in.; results for thickness of 1 in. and 3 in. are analyzed. In each case, three equally spaced support arcs are considered.

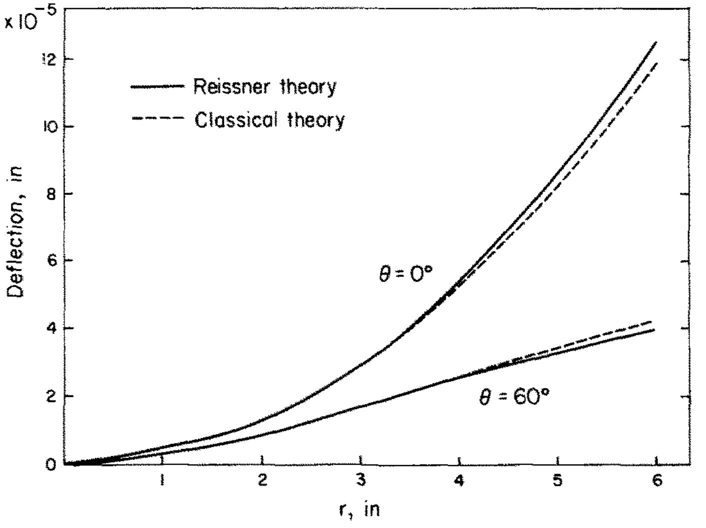

FIG. 3. Deflection vs. radius—1 in. thick plate supported by three equally spaced 30° support arcs on the periphery.

Figure 3 shows a plot of deflection versus radius for a 6-in.-radius disk supported on 30° arcs. Deflections are shown for radii oriented at 0 and 60°. For the thickness-to-diameter ratio considered, $\frac{1}{12}$, it can be seen that very little difference exists between results obtained from the Reissner and classical theories.

Figure 4 shows deflection vs. angle on the outer periphery of the same disk. Results are shown for both 30 and 60° support arc lengths. Once again, it can be seen that very little difference exists between results obtained through use of the Reissner and classical theories. As expected, deflections are reduced by use of longer arc supports.

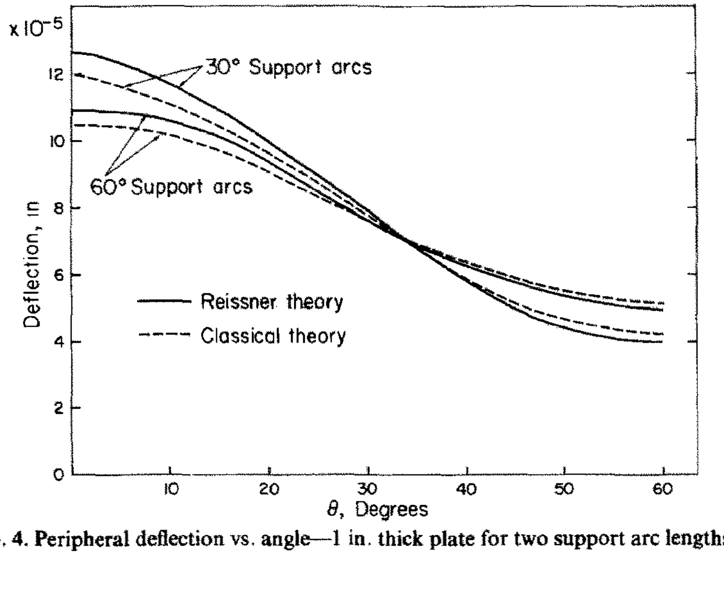

FIG. 4. Peripheral deflection vs. angle—1 in. thick plate for two support arc lengths.

Figure 5 shows a plot of deflection vs. angle for a disk of 6 in. radius having a thickness of 3 in., and supported on 30° arcs. Deflections are shown for radii oriented at 0 and 60°. For the thickness-to-diameter ratio considered, $\frac{1}{4}$, it can be seen that a significant difference exists between results obtained through use of the Reissner and classical theories.

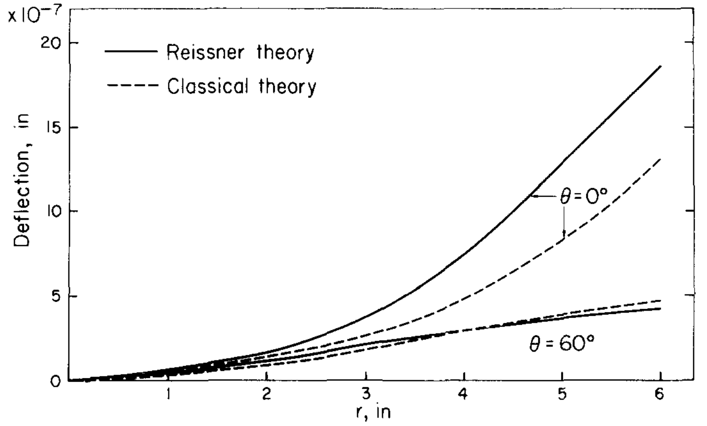

FIG. 5. Deflection vs. radius—3 in. thick plate supported by three equally spaced 30° support arcs on the periphery.

Figure 6 shows deflection vs. angle on the outer periphery of the same 3-in.-thick disk. Results are shown for both 30 and 60° support arc lengths. Once again, it can be seen that a large difference exists between results obtained from the Reissner and classical theories.

It is clear that for plates having thickness-to-diameter ratios of greater than approximately one-tenth, shearing deformations can contribute significantly to total deflection, and hence should not be neglected.

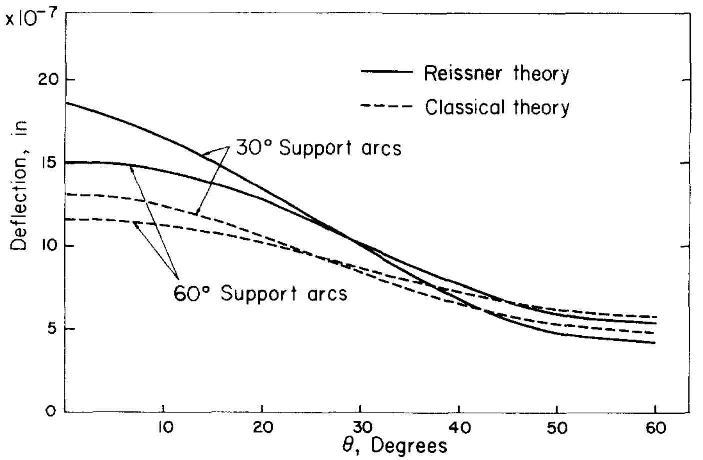

FIG. 6. Peripheral deflection vs. angle—3 in. thick plate for two support arc lengths.

Figure 7 shows a plot of shear stress resultant, V_r, vs. radius, and Fig. 8 a plot of bending couple, M_r, vs. radius, for a disk of 6 in. radius having a thickness of 3 in., and supported on 60° arcs. Results are shown for radii oriented at 0 and 60°. It is clear that for plates having thickness-to-diameter ratios of greater than about one-tenth, shear effects contribute significantly to stresses, and hence should not be neglected.

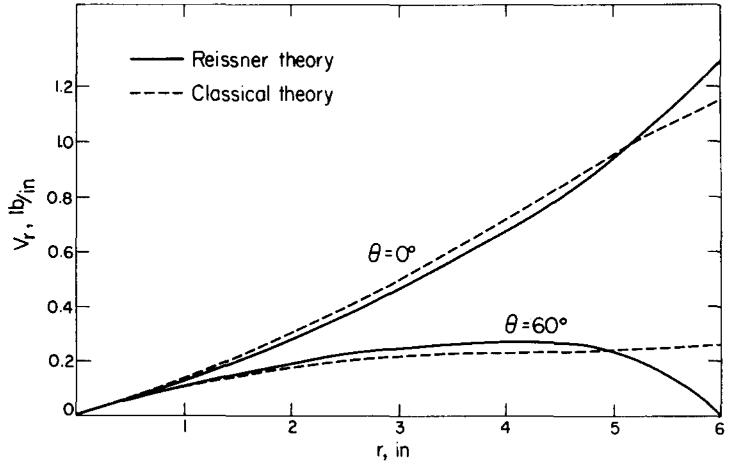

FIG. 7. Transverse shear stress resultant vs. radius—3 in. thick plate supported by three equally spaced 60° support arcs on the periphery.

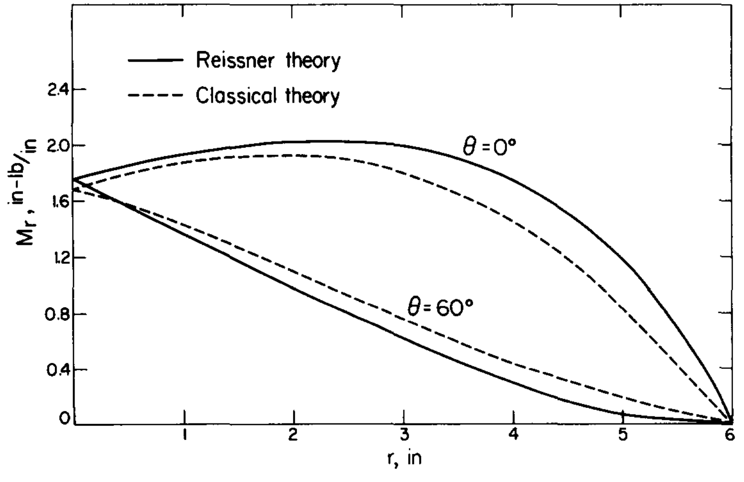

FIG. 8. Bending couple vs. radius—3 in. thick plate supported by three equally spaced 60° support arcs on the periphery.

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Абстракт—В работе даются решенмя для изгибов под влиянием собственного веса, моментов и сдвигов в толстых, горизонтально расположенных, круглых плитах, имеющих по окружности перерывные, закрывленные опоры. Используется теория предложенная Рейсснером для толстых плит, которая заключает деформации сдвига. Результаты сравниваются с такими же результатами для классической теории пластинок. Находится, что для плит, обладающих отношением толщины к дияметру, являющимся большим приблизительно одной десятой, эффекты сдвига могут значительно влиять на полные озгибы и напряжения, которыми нельзя пренебрегать. Даются численные результаты, которые обсуждаются подробно.